

Area Products for Taub-NUT and Kerr-Taub-NUT Space-time

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Abstract

We examine properties of the inner and outer horizon thermodynamics of Taub-NUT(Newman-Uni-Tamburino) and Kerr-Taub-NUT black holes in four dimensional gravity theories. We compare and contrasted these properties with the properties of Reissner Nordström black hole and Kerr black hole. We focus on “area product”, “entropy product” , “irreducible mass product” of the event horizon and Cauchy horizons of the said black hole. We find that these products have no beautiful quantization features. Nor does it has any mass-independence(universal) properties. We also show that the *First law* of black hole thermodynamics and *Smarr-Gibbs-Duhem* relation do not hold for Taub-NUT and Kerr-Taub-NUT black hole. This is happening due to the explicitly presence of the NUT parameter. The black hole *mass formula* and *Christodoulou-Ruffini mass formula* for Taub-NUT and Kerr-Taub-NUT black holes are also computed.

1 Introduction

Perhaps the most remarkable and beautiful analogy between the four laws of black hole physics and the laws of thermodynamics were first discovered in the early 1970s [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. Particularly, the Bekenstein-Hawking area-entropy relation which is described by the well known formula:

$$\mathcal{S}_+ = \frac{\mathcal{A}_+}{4} \quad (1)$$

where, \mathcal{S}_+ is the Bekenstein-Hawking entropy (in units in which $G = \hbar = c = k = 1$) and \mathcal{A}_+ is the area of the event horizon(\mathcal{H}^+).

Another striking mathematical correspondence was established by Hawking’s discovery [4] that black holes radiate as perfect black bodies at

$$T_+ = \frac{\kappa_+}{2\pi} \quad (2)$$

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where T_+ is the Hawking temperature computed at the \mathcal{H}^+ and κ_+ denotes the surface gravity of the black hole computed at the \mathcal{H}^+ .

It is now well known by fact that certain black holes possess inner horizon or Cauchy horizon (\mathcal{H}^-) in addition with the outer horizon or event horizon. So, when the above geometric quantity holds for \mathcal{H}^+ , there might have indication that these relations also hold for \mathcal{H}^- . This idea first comes from the fact that the product of the event horizon (\mathcal{H}^+) and the Cauchy horizon's (\mathcal{H}^-) areas of any asymptotically flat black hole admitting a smooth extremal limit, seems to depend only on the quantized charges and is independent of the ADM (Arnowitt-Deser-Misner) mass of the space-time [12] and see also [13, 14, 15, 16, 17, 18, 19, 20].

Curir[21] first showed that the first law of black hole thermodynamics holds on the Cauchy horizon as well as event horizon:

$$d\mathcal{M} = T_{\pm}d\mathcal{S}_{\pm} + \Omega_{\pm}dJ. \quad (3)$$

The same author also calculated the “area product” and “area sum” of Kerr BH for the interpretation of the spin entropy of the inner horizon. That is

$$\mathcal{A}_+\mathcal{A}_- = 64\pi^2 J^2. \quad (4)$$

and

$$\mathcal{A}_+ + \mathcal{A}_- = 16\pi\mathcal{M}^2. \quad (5)$$

He also showed that the sum of entropy of both the horizons (\mathcal{H}^{\pm}) is equal to the entropy of a Schwarzschild black hole [22]. That is

$$\mathcal{S}_+ + \mathcal{S}_- = 4\pi\mathcal{M}^2. \quad (6)$$

In a very recent work [23], the authors have investigated an entropy sum relation of BHs in four dimensional AdS space-time in asymptotically background. They showed that the entropy sum and area sum depends on the cosmological parameter and also does not depend on the ADM mass (\mathcal{M}) and charge (Q) parameter.

It is now well known that the Cauchy horizon has the property of being a null surface of infinite blue-shift, while the event horizon is an infinite red-shift surface [25]. It is also true that the inner horizons of both the Reissner Nordström (RN) and Kerr BHs are highly unstable due to the exterior perturbation. That means, the energy-momentum tensor associated with various massive and massless test fields diverges on the Cauchy horizon [24, 26]. In another work, Poisson and Israel have proved that when a RN BH is perturbed its inner horizon becomes a singularity of infinite space-time curvature -which is known as mass inflation singularity [27].

Thus contrary to the outer horizon, there might have relevance of the inner horizon in the BH thermodynamics. Therefore the Cauchy horizon takes now an important position

in BH thermodynamics to explain the Bekenstein-Hawking entropy and different thermodynamic products. This is why we study the following thermodynamic products like area product, entropy product and irreducible mass product of the event horizon and Cauchy horizons. We consider here Taub-NUT BH and Kerr-Taub-NUT BH. We show whether the first law of BH thermodynamics does hold for Taub-NUT BH and Kerr-Taub-NUT BH. This is a crucial point to investigate this properties due to the inclusion of the NUT parameter [28] or gravito-magnetic mass. We also compute the Smarr's mass formula[29] and Christodoulou-Ruffini mass formula for the above mentioned BH. This is the main purpose of this paper.

The plan of the paper is as follows. In Sec.(2), we describe various thermodynamic products for Taub-NUT BH. In this Sec., there are two sub-section. In first sub-section, we derive the mass formula for Taub-NUT BH. In second sub-section, we derive the Christodoulou-Ruffini mass formula for Taub-NUT BH. In Sec.(3), we discuss various thermodynamic products for Kerr-Taub-NUT BH. In the first sub-section, we describe the Smarr's mass formula for Kerr-Taub-NUT BH. In the second sub-section, we derive the Christodoulou-Ruffini mass formula for Kerr-Taub-NUT BH. Finally, we conclude in Sec.(4).

2 Area Product and Area sum of Taub- NUT BH:

First we consider the Taub- NUT BH, it is a stationary, spherically-symmetric vacuum solution of Einstein equation with the NUT parameter (n). This NUT charge or dual mass has an intrinsic feature in Einstein's general relativity and which is the gravitational analogue of a magnetic monopole in Maxwell's electrodynamics [34].

Thus the metric is given by [31, 36]

$$ds^2 = -U(r) (dt + 2n \cos \theta d\phi)^2 + \frac{dr^2}{U(r)} + (r^2 + n^2) (d\theta^2 + \sin^2 \theta d\phi^2) . \quad (7)$$

$$U(r) = 1 - \frac{2(\mathcal{M}r + n^2)}{r^2 + n^2} \quad (8)$$

Here, \mathcal{M} represents the gravito-electric mass or ADM mass and n represents the gravito-magnetic mass or dual mass or magnetic mass of the spacetime. The idea of dual mass could be found in [35] in details. The black hole horizons being given by

$$r_{\pm} = \mathcal{M} \pm \sqrt{\mathcal{M}^2 + n^2} \text{ and } r_+ > r_- \quad (9)$$

Here, r_+ is called event horizon (\mathcal{H}^+) or outer horizon and r_- is called Cauchy horizon (\mathcal{H}^-) or inner horizon. The area of both the horizons are

$$\mathcal{A}_{\pm} = \int \int \sqrt{g_{\theta\theta}g_{\phi\phi}} d\theta d\phi = 4\pi(r_{\pm}^2 + n^2) \quad (10)$$

$$= 8\pi \left[(\mathcal{M}^2 + n^2) \pm \mathcal{M}\sqrt{\mathcal{M}^2 + n^2} \right] . \quad (11)$$

The area products can be computed to be

$$\mathcal{A}_+ \mathcal{A}_- = (8\pi n)^2 (\mathcal{M}^2 + n^2) . \quad (12)$$

It shows that the product is not mass independent. Thus the conjectured “area product is universal” does not hold for Taub-NUT spacetime.

Now the area sum could be obtained as,

$$\mathcal{A}_+ + \mathcal{A}_- = 16\pi (\mathcal{M}^2 + n^2) . \quad (13)$$

It implies that area sum does depend on the mass, and is therefore not universal.

The black hole entropy of \mathcal{H}^\pm is proportional to the surface area of \mathcal{H}^\pm and is given by (in units in which $G = \hbar = c = 1$)

$$\mathcal{S}_\pm = \frac{\mathcal{A}_\pm}{4} = \pi(r_\pm^2 + n^2) = 2\pi (\mathcal{M}r_\pm + n^2) . \quad (14)$$

Their product is calculated to be

$$\mathcal{S}_+ \mathcal{S}_- = (2\pi n)^2 (\mathcal{M}^2 + n^2) . \quad (15)$$

and their sum is

$$\mathcal{S}_+ + \mathcal{S}_- = 4\pi [\mathcal{M}^2 + n^2] . \quad (16)$$

It indicates that entropy product and entropy sum are not universal. The surface gravity computed at the \mathcal{H}^\pm is

$$\kappa_\pm = \frac{r_\pm - r_\mp}{2(r_\pm^2 + n^2)} = \frac{r_\pm - r_\mp}{4(\mathcal{M}r_\pm + n^2)} \text{ and } \kappa_+ > \kappa_- . \quad (17)$$

Since the black hole temperature of \mathcal{H}^\pm is proportional to the surface gravity of \mathcal{H}^\pm and is given by

$$T_\pm = \frac{\kappa_\pm}{2\pi} = \frac{r_\pm - r_\mp}{4\pi(r_\pm^2 + n^2)} = \frac{r_\pm - r_\mp}{8\pi(\mathcal{M}r_\pm + n^2)} . \quad (18)$$

It should be noted that $T_+ > T_-$. In the limit $\mathcal{M} = 0$, we obtain the result of massless Taub-NUT BH.

Now we calculate various thermodynamic quantities of the Reissner Nordström BH in comparison with the Taub-NUT BH which are tabulated in the below.

Parameter	RN BH	Taub-NUT BH
$r_{\pm}:$	$\mathcal{M} \pm \sqrt{\mathcal{M}^2 - Q^2}$	$\mathcal{M} \pm \sqrt{\mathcal{M}^2 + n^2}$
$\sum r_i:$	$2\mathcal{M}$	$2\mathcal{M}$
$\prod r_i:$	Q^2	$-n^2$
$\mathcal{A}_{\pm}:$	$4\pi(2\mathcal{M}r_{\pm} - Q^2)$	$8\pi(\mathcal{M}r_{\pm} + n^2)$
$\sum \mathcal{A}_i:$	$8\pi(2\mathcal{M}^2 - Q^2)$	$16\pi(\mathcal{M}^2 + n^2)$
$\prod \mathcal{A}_i:$	$(4\pi Q^2)^2$	$(8\pi n)^2[\mathcal{M}^2 + n^2]$
$\mathcal{S}_{\pm}:$	$\pi(2\mathcal{M}r_{\pm} - Q^2)$	$2\pi(\mathcal{M}r_{\pm} + n^2)$
$\sum \mathcal{S}_i:$	$2\pi(2\mathcal{M}^2 - Q^2)$	$4\pi(\mathcal{M}^2 + n^2)$
$\prod \mathcal{S}_i:$	$\pi^2 Q^4$	$(2\pi n)^2[\mathcal{M}^2 + n^2]$
$\kappa_{\pm}:$	$\frac{r_{\pm} - r_{\mp}}{2(2\mathcal{M}r_{\pm} - Q^2)}$	$\frac{r_{\pm} - r_{\mp}}{4(\mathcal{M}r_{\pm} + n^2)}$
$\sum \kappa_i:$	$\frac{4\mathcal{M}(Q^2 - \mathcal{M}^2)}{Q^4}$	$-\frac{\mathcal{M}}{n^2}$
$\prod \kappa_i:$	$\frac{Q^2 - \mathcal{M}^2}{Q^4}$	$-\frac{1}{4n^2}$
$T_{\pm}:$	$\frac{r_{\pm} - r_{\mp}}{4\pi r_{\pm}^2}$	$\frac{r_{\pm} - r_{\mp}}{4\pi(r_{\pm}^2 + n^2)}$
$\sum T_i:$	$\frac{2\mathcal{M}(Q^2 - \mathcal{M}^2)}{\pi Q^4}$	$-\frac{\mathcal{M}}{2\pi n^2}$
$\prod T_i:$	$\frac{(Q^2 - \mathcal{M}^2)}{4\pi^2(Q^4)}$	$-\frac{1}{(4\pi n)^2}$
$\mathcal{M}_{irr,\pm}:$	$\sqrt{\frac{\mathcal{A}_{\pm}}{16\pi}}$	$\sqrt{\frac{\mathcal{A}_{\pm}}{16\pi}}$
$\sum \mathcal{M}_{irr}^2:$	$\mathcal{M}^2 - \frac{Q^2}{2}$	$\mathcal{M}^2 + n^2$
$\prod \mathcal{M}_{irr}:$	$\frac{Q^2}{4}$	$\sqrt{\frac{n^2(\mathcal{M}^2 + n^2)}{4}}$

2.1 The Mass Formula for Taub NUT Spacetime:

Smarr[29] first showed that mass can be expressed as a function of area, angular momentum and charge for Kerr-Newman black hole. On the other hand, Hawking[9] pointed out that the BH surface area can never decreases. Therefore the BH surface area is indeed a constant quantity over the \mathcal{H}^{\pm} . For Taub-NUT spacetime, this is given by Eq.(11). Inverting this Eq. one can obtain for Taub-NUT BH, the mass as a function of area of both the horizons(\mathcal{H}^{\pm}) and NUT parameter n :

$$\mathcal{M}^2 = \frac{1}{(\mathcal{A}_{\pm} - 4\pi n^2)} \left[\frac{\mathcal{A}_{\pm}^2}{16\pi} - n^2(\mathcal{A}_{\pm} - 4\pi n^2) \right]. \quad (19)$$

Let us calculate the mass differential $d\mathcal{M}$ to define the two physical quantities in terms of inner and outer horizon.

$$d\mathcal{M} = \Gamma_{\pm} d\mathcal{A}_{\pm} + \Phi_{\pm}^n dn. \quad (20)$$

where

$$\Gamma_{\pm} = \frac{\mathcal{A}_{\pm}(\mathcal{A}_{\pm} - 8\pi n^2)}{32\pi\mathcal{M}(\mathcal{A}_{\pm} - 4\pi n^2)^2} \quad (21)$$

$$\Phi_{\pm}^n = \frac{n(16\pi n^2\mathcal{A}_{\pm} - \frac{3}{2}\mathcal{A}_{\pm}^2 - 32\pi^2 n^4)}{2\mathcal{M}(\mathcal{A}_{\pm} - 4\pi n^2)^2} . \quad (22)$$

where,

Γ_{\pm} = Effective surface tension of \mathcal{H}^+ and \mathcal{H}^-

Φ_{\pm}^n = Electromagnetic potentials or Taub-NUT potentials of \mathcal{H}^{\pm} for NUT charge

Using the Euler's theorem on homogenous functions to \mathcal{M} of degree $\frac{1}{2}$ in (\mathcal{A}_{\pm}, n^2) , one can derive the mass for Taub-NUT BH can be expressed as

$$\mathcal{M} = 2\Gamma_{\pm}\mathcal{A}_{\pm} + \Phi_{\pm}^n n . \quad (23)$$

Remarkably, Γ_{\pm} and Φ_{\pm}^n can be defined and are constant over the \mathcal{H}^+ and \mathcal{H}^- for any stationary space-time.

Generally, combining the mass differential Eq. (20) with the first law leads to the Smarr-Gibbs-Duhem relation. We can see easily from Eqs. (20) and (23), such an equation does not hold for Taub-NUT spacetime because

$$\Gamma_{\pm} = \frac{\partial\mathcal{M}}{\partial\mathcal{A}_{\pm}} \neq \frac{\kappa_{\pm}}{8\pi} \quad (24)$$

An important point should be noted here that *the first law of BH thermodynamics* do not satisfied for the Taub-NUT spacetime.

In the case of RN BH, we may noted that the mass formula is given by

$$\mathcal{M}^2 = \frac{\mathcal{A}_{\pm}}{16\pi} + \frac{\pi Q^4}{\mathcal{A}_{\pm}} + \frac{Q^2}{2} . \quad (25)$$

Again the mass can be expressed as in a simple bilinear form:

$$\mathcal{M} = 2\Gamma_{\pm}\mathcal{A}_{\pm} + \Phi_{\pm}Q . \quad (26)$$

where,

$$\Gamma_{\pm} = \frac{\partial\mathcal{M}}{\partial\mathcal{A}_{\pm}} = \frac{\kappa_{\pm}}{8\pi} . \quad (27)$$

and

$$\Phi_{\pm} = \frac{\partial\mathcal{M}}{\partial Q} = \frac{1}{\mathcal{M}} \left(\frac{Q}{2} + \frac{2\pi Q^3}{\mathcal{A}_{\pm}} \right) . \quad (28)$$

Thus the Smarr-Gibbs-Duhem relation can be obtained as

$$\frac{\mathcal{M}}{2} = T_{\pm}\mathcal{S}_{\pm} + \frac{\Phi_{\pm}Q}{2} . \quad (29)$$

It can be seen that both Smarr-Gibbs-Duhem relation and First law of BH thermodynamics satisfied in case of RN BH.

2.2 Christodoulou's Irreducible Mass for Taub-NUT Space-time:

Christodoulou has shown that the irreducible mass of the BH, \mathcal{M}_{irr} is proportional to the square root of the black hole's surface area. He has emphasized that the irreducible mass of a Kerr BH can be expressed as in terms of event horizon area. We have already been suggested that for KN BH [30] the irreducible mass could be expressed as in terms of both outer and inner horizon area. That means the irreducible mass can be defined as

$$\mathcal{M}_{irr,\pm} = \sqrt{\frac{\mathcal{A}_{\pm}}{16\pi}}. \quad (30)$$

where, $+$ indicates for \mathcal{H}^+ and $-$ indicates for \mathcal{H}^- . Thus for Taub-NUT black hole this could be written as

$$\mathcal{M}_{irr,\pm} = \frac{\sqrt{r_{\pm}^2 + n^2}}{2}. \quad (31)$$

The product of the inner irreducible mass and outer irreducible mass for Taub-NUT space-time is given by

$$\mathcal{M}_{irr,+}\mathcal{M}_{irr,-} = \sqrt{n^2(\mathcal{M}^2 + n^2)}. \quad (32)$$

and therefore depends on the mass of the black hole.

Another interesting formula i.e. the Christodoulou-Ruffini mass formula for Taub-NUT space-time in term of its irreducible mass and NUT parameter n could be derived as

$$\mathcal{M}^2 = \left[(\mathcal{M}_{irr,\pm})^2 - n^2 \left(1 - \frac{n^2}{4(\mathcal{M}_{irr,\pm})^2} \right) \right] \times \left(1 - \frac{n^2}{4(\mathcal{M}_{irr,\pm})^2} \right)^{-1}. \quad (33)$$

When the NUT parameter goes to zero we get the $\mathcal{M} = \mathcal{M}_{irr}$ for Schwarzschild space-time [2].

For our record, we may write the Christodoulou-Ruffini mass formula for RN BH:

$$\mathcal{M} = \mathcal{M}_{irr,\pm} + \frac{Q^2}{4\mathcal{M}_{irr,\pm}}. \quad (34)$$

3 Area Product and Area Sum for Kerr-Taub-NUT BH:

Now we turn to the Kerr Taub-NUT black hole. It is also a stationary, axially symmetric vacuum solution of Einstein equation with Kerr parameter (a) and NUT parameter (n). When $n = 0$, we obtain the Kerr geometry.

The metric [32, 33] is described in Boyer-Lindquist like spherical coordinates (t, r, θ, ϕ) :

$$ds^2 = -\frac{\Delta}{\Sigma} [dt - \chi d\phi]^2 + \frac{\sin^2 \theta}{\Sigma} [(r^2 + a^2 + n^2) d\phi - a dt]^2 + \Sigma \left[\frac{dr^2}{\Delta} + d\theta^2 \right]. \quad (35)$$

where,

$$a \equiv \frac{J}{\mathcal{M}}, \quad \Sigma \equiv r^2 + (n + a \cos \theta)^2 \quad (36)$$

$$\Delta \equiv r^2 - 2\mathcal{M}r + a^2 - n^2 \quad (37)$$

$$\chi \equiv a \sin^2 \theta - 2n \cos \theta. \quad (38)$$

Thus the metric is completely determined by the three parameters i.e., the mass (\mathcal{M}), angular momentum ($J = a\mathcal{M}$) and gravito-magnetic monopole or NUT parameter (n) or magnetic mass. The spacetime has no curvature singularity, there are conical singularities on its axis of symmetry that result in the gravito-magnetic analogue of Dirac's string quantization condition [37] and also leading to Dirac-Misner string singularities at poles $\theta = 0, \pi$. The conical singularities can be removed by imposing an appropriate periodicity condition on the time coordinate. It should be noted that the presence of the NUT parameter in the spacetime destroys its asymptotic structure making it asymptotically non-flat.

The radius of the outer and inner horizon are given by

$$r_{\pm} = \mathcal{M} \pm \sqrt{\mathcal{M}^2 + n^2 - a^2} \quad (39)$$

The inner and outer horizons area are

$$\mathcal{A}_{\pm} = \int \int \sqrt{g_{\theta\theta}g_{\phi\phi}} d\theta d\phi = 4\pi(r_{\pm}^2 + a^2 + n^2) \quad (40)$$

$$= 8\pi \left[(\mathcal{M}^2 + n^2) \pm \mathcal{M}\sqrt{\mathcal{M}^2 + n^2 - a^2} \right]. \quad (41)$$

The product of $\mathcal{A}_+\mathcal{A}_-$ is given by

$$\mathcal{A}_+\mathcal{A}_- = (8\pi)^2 (J^2 + n^2(\mathcal{M}^2 + n^2)). \quad (42)$$

and the sum of $\mathcal{A}_+ + \mathcal{A}_-$ is given by

$$\mathcal{A}_+ + \mathcal{A}_- = 16\pi [\mathcal{M}^2 + n^2]. \quad (43)$$

The area product and area sum shown not to be universal for Kerr-Taub-NUT black hole.

The Bekenstein-Hawking [6, 9] entropy associated with this black hole reads

$$\mathcal{S}_{\pm} = \frac{\mathcal{A}_{\pm}}{4} = \pi(r_{\pm}^2 + a^2 + n^2) = 2\pi (\mathcal{M}r_{\pm} + n^2). \quad (44)$$

Their product can be derived as

$$\mathcal{S}_+ \mathcal{S}_- = (2\pi)^2 (J^2 + n^2(\mathcal{M}^2 + n^2)) . \quad (45)$$

and their sum reads

$$\mathcal{S}_+ + \mathcal{S}_- = 4\pi [\mathcal{M}^2 + n^2] . \quad (46)$$

It implies that entropy product and entropy sum both are depends on mass.

The surface gravity calculated at \mathcal{H}^\pm to be

$$\kappa_\pm = \frac{r_\pm - r_\mp}{2(r_\pm^2 + a^2 + n^2)} = \frac{r_\pm - r_\mp}{2(2\mathcal{M}r_\pm + 2n^2)} . \quad (47)$$

and the BH temperature or Hawking temperature of \mathcal{H}^\pm reads as

$$T_\pm = \frac{\kappa_\pm}{2\pi} = \frac{r_\pm - r_\mp}{4\pi(r_\pm^2 + a^2 + n^2)} . \quad (48)$$

Now, we shall compute various thermodynamic quantities for KN BH, Kerr BH in comparison with Kerr-Taub-NUT(KTN) BH.

Parameter	KN BH	Kerr BH	KTN BH
$r_{\pm}:$	$\mathcal{M} \pm \sqrt{\mathcal{M}^2 - a^2 - Q^2}$	$M \pm \sqrt{\mathcal{M}^2 - a^2}$	$\mathcal{M} \pm \sqrt{\mathcal{M}^2 + n^2 - a^2}$
$\sum r_i:$	$2\mathcal{M}$	$2\mathcal{M}$	$2\mathcal{M}$
$\prod r_i:$	$a^2 + Q^2$	a^2	$a^2 - n^2$
$\mathcal{A}_{\pm}:$	$4\pi(2\mathcal{M}r_{\pm} - Q^2)$	$8\pi\mathcal{M}r_{\pm}$	$8\pi(\mathcal{M}r_{\pm} + n^2)$
$\sum \mathcal{A}_i:$	$8\pi(2\mathcal{M}^2 - Q^2)$	$16\pi\mathcal{M}^2$	$16\pi(\mathcal{M}^2 + n^2)$
$\prod \mathcal{A}_i:$	$(8\pi)^2 \left(J^2 + \frac{Q^4}{4} \right)$	$(8\pi J)^2$	$(8\pi)^2 [J^2 + n^2(\mathcal{M}^2 + n^2)]$
$S_{\pm}:$	$\pi(2\mathcal{M}r_{\pm} - Q^2)$	$2\pi\mathcal{M}r_{\pm}$	$2\pi(\mathcal{M}r_{\pm} + n^2)$
$\sum \mathcal{S}_i:$	$2\pi(2\mathcal{M}^2 - Q^2)$	$4\pi\mathcal{M}^2$	$4\pi(\mathcal{M}^2 + n^2)$
$\prod \mathcal{S}_i:$	$(2\pi)^2 \left(J^2 + \frac{Q^4}{4} \right)$	$(2\pi J)^2$	$(2\pi)^2 [J^2 + n^2(\mathcal{M}^2 + n^2)]$
$\kappa_{\pm}:$	$\frac{r_{\pm} - r_{\mp}}{2(2\mathcal{M}r_{\pm} - Q^2)}$	$\frac{r_{\pm} - r_{\mp}}{4\mathcal{M}r_{\pm}}$	$\frac{r_{\pm} - r_{\mp}}{4(\mathcal{M}r_{\pm} + n^2)}$
$\sum \kappa_i:$	$\frac{4\mathcal{M}(a^2 + Q^2 - \mathcal{M}^2)}{(4J^2 + Q^4)}$	$\frac{a^2 - \mathcal{M}^2}{aJ}$	$\frac{\mathcal{M}(a^2 - \mathcal{M}^2 - n^2)}{[J^2 + n^2(\mathcal{M}^2 + n^2)]}$
$\prod \kappa_i:$	$\frac{a^2 + Q^2 - \mathcal{M}^2}{(4J^2 + Q^4)}$	$\frac{a^2 - \mathcal{M}^2}{4J^2}$	$\frac{(a^2 - \mathcal{M}^2 - n^2)}{4[J^2 + n^2(\mathcal{M}^2 + n^2)]}$
$T_{\pm}:$	$\frac{r_{\pm} - r_{\mp}}{4\pi(r_{\pm}^2 + a^2)}$	$\frac{r_{\pm} - r_{\mp}}{4\pi(r_{\pm}^2 + a^2)}$	$\frac{r_{\pm} - r_{\mp}}{4\pi(r_{\pm}^2 + a^2)}$
$\sum T_i:$	$\frac{2\mathcal{M}(a^2 + Q^2 - \mathcal{M}^2)}{\pi(4J^2 + Q^4)}$	$\frac{a^2 - \mathcal{M}^2}{2\pi aJ}$	$\frac{\mathcal{M}(a^2 - \mathcal{M}^2 - n^2)}{2\pi[J^2 + n^2(\mathcal{M}^2 + n^2)]}$
$\prod T_i:$	$\frac{(a^2 + Q^2 - \mathcal{M}^2)}{4\pi^2(4J^2 + Q^4)}$	$\frac{a^2 - \mathcal{M}^2}{(4\pi J)^2}$	$\frac{(a^2 - \mathcal{M}^2 - n^2)}{(4\pi)^2[J^2 + n^2(\mathcal{M}^2 + n^2)]}$
$\mathcal{M}_{irr,\pm}:$	$\sqrt{\frac{2\mathcal{M}r_{\pm} - Q^2}{4}}$	$\sqrt{\frac{\mathcal{M}r_{\pm}}{2}}$	$\sqrt{\frac{\mathcal{M}r_{\pm} + n^2}{2}}$
$\sum \mathcal{M}_{irr}^2:$	$\mathcal{M}^2 - \frac{Q^2}{2}$	\mathcal{M}^2	$\mathcal{M}^2 + n^2$
$\prod \mathcal{M}_{irr}:$	$\sqrt{\frac{J^2 + \frac{Q^4}{4}}{4}}$	$\frac{J}{2}$	$\sqrt{\frac{J^2 + n^2(\mathcal{M}^2 + n^2)}{4}}$
$\Omega_{\pm}:$	$\frac{a}{2\mathcal{M}r_{\pm} - Q^2}$	$\frac{a}{2\mathcal{M}r_{\pm}}$	$\frac{a}{2\mathcal{M}r_{\pm} + 2n^2}$
$\sum \Omega_i:$	$\frac{2a(2\mathcal{M}^2 - Q^2)}{4J^2 + Q^4}$	$\frac{1}{a}$	$\frac{a(\mathcal{M}^2 + n^2)}{[J^2 + n^2(\mathcal{M}^2 + n^2)]}$
$\prod \Omega_i:$	$\frac{a^2}{4J^2 + Q^4}$	$\frac{1}{4\mathcal{M}^2}$	$\frac{a^2}{4[J^2 + n^2(\mathcal{M}^2 + n^2)]}$

3.1 The Mass Formula for Kerr-Taub-NUT Spacetime:

Now it is straightforward calculation to compute the surface area of Kerr-Taub-NUT BH in terms of outer and inner horizon:

$$\mathcal{A}_{\pm} = 8\pi \left(\mathcal{M}^2 + n^2 \pm \sqrt{\mathcal{M}^4 - J^2 + n^2\mathcal{M}^2} \right). \quad (49)$$

It should be noted that the black hole surface area is indeed constant over the \mathcal{H}^{\pm} . The above equation could be inverted to give

$$\mathcal{M}^2 = \frac{1}{(\mathcal{A}_{\pm} - 4\pi n^2)} \left[\frac{\mathcal{A}_{\pm}^2}{16\pi} + 4\pi J^2 - n^2(\mathcal{A}_{\pm} - 4\pi n^2) \right]. \quad (50)$$

which shows that the black hole mass can be expressed as in terms of both the area of \mathcal{H}^+ and \mathcal{H}^- .

The mass differential in terms of three physical invariants of both \mathcal{H}^+ and \mathcal{H}^- , for Kerr-Taub-NUT space-time could be derived as

$$d\mathcal{M} = \Gamma_{\pm} d\mathcal{A}_{\pm} + \Omega_{\pm} dJ + \Phi_{\pm}^n dn. \quad (51)$$

where

$$\Gamma_{\pm} = \frac{1}{2\mathcal{M}(\mathcal{A}_{\pm} - 4\pi n^2)^2} \left(\frac{\mathcal{A}_{\pm}^2}{16\pi} - 4\pi J^2 - \frac{n^2 \mathcal{A}_{\pm}}{2} \right) \quad (52)$$

$$\Omega_{\pm} = \frac{4\pi J}{\mathcal{M}(\mathcal{A}_{\pm} - 4\pi n^2)} \quad (53)$$

$$\Phi_{\pm}^n = \frac{(16\pi n^3 \mathcal{A}_{\pm} - \frac{3}{2} n \mathcal{A}_{\pm}^2 - 32\pi^2 n^5 + 32n\pi J^2)}{2\mathcal{M}(\mathcal{A}_{\pm} - 4\pi n^2)^2}. \quad (54)$$

where,

Γ_{\pm} = Effective surface tension of \mathcal{H}^+ and \mathcal{H}^-

Ω_{\pm} = Angular velocity of \mathcal{H}^{\pm}

Φ_{\pm}^n = Electromagnetic potentials or Taub-NUT potentials of \mathcal{H}^{\pm} for NUT charge

Since the mass of black holes are homogenous of order $\frac{1}{2}$ in the variables $(\mathcal{A}_{\pm}, J, n^2)$, therefore using the Euler's theorem on homogenous function to \mathcal{M} one obtains,

$$\mathcal{M} = 2\Gamma_{\pm} \mathcal{A}_{\pm} + 2J\Omega_{\pm} + \Phi_{\pm}^n n. \quad (55)$$

It implies that the \mathcal{M} can be expressed in terms of these quantities both for \mathcal{H}^{\pm} as a bilinear form. It may be noted that Γ_{\pm} , Ω_{\pm} and Φ_{\pm}^n could be defined and are indeed constant on the \mathcal{H}^+ and \mathcal{H}^- for any stationary, axially symmetric spacetime.

Again the $d\mathcal{M}$ is perfect differential, one can freely defined any path of integration in $(\mathcal{A}_{\pm}, J, n)$ space. Thus the surface energy $\mathcal{E}_{s,\pm}$ for \mathcal{H}^{\pm} can be defined as

$$\mathcal{E}_{s,\pm} = \int_0^{\mathcal{A}_{\pm}} \Gamma(\tilde{\mathcal{A}}_{\pm}, 0, 0) d\tilde{\mathcal{A}}_{\pm}; \quad (56)$$

The rotational energy for \mathcal{H}^{\pm} can be defined by

$$\mathcal{E}_{r,\pm} = \int_0^J \Omega_{\pm}(\mathcal{A}_{\pm}, \tilde{J}, 0) d\tilde{J}, \mathcal{A}_{\pm} \text{ fixed}; \quad (57)$$

and finally the electromagnetic energy for \mathcal{H}^{\pm} due to the NUT charge n is given by

$$\mathcal{E}_{em,\pm} = \int_0^n \Phi_{\pm}(\mathcal{A}_{\pm}, J, \tilde{n}) d\tilde{n}, \mathcal{A}_{\pm}, J \text{ fixed}; \quad (58)$$

Now combining the mass differential Eq. (51) and the first law gives the Smarr-Gibbs-Duhem relation. We see that from Eqs. (51) and (55), such an equation does not hold for Kerr-Taub-NUT BH because

$$\Gamma_{\pm} = \frac{\partial \mathcal{M}}{\partial \mathcal{A}_{\pm}} \neq \frac{\kappa_{\pm}}{8\pi} = \frac{T_{\pm}}{4} \quad (59)$$

We also observed that *the first law of BH thermodynamics* does not hold for Kerr-Taub-NUT BH.

Whereas, the BH mass or ADM mass can be expressed as in terms of area of both horizons \mathcal{H}^{\pm} for Kerr BH reads

$$\mathcal{M}^2 = \frac{\mathcal{A}_{\pm}}{16\pi} + \frac{4\pi J^2}{\mathcal{A}_{\pm}}. \quad (60)$$

Again the mass can be expressed as in terms of a simple bilinear form:

$$\mathcal{M} = 2\Gamma_{\pm}\mathcal{A}_{\pm} + 2J\Omega_{\pm}. \quad (61)$$

where,

$$\Gamma_{\pm} = \frac{\partial \mathcal{M}}{\partial \mathcal{A}_{\pm}} = \frac{\kappa_{\pm}}{8\pi}. \quad (62)$$

and

$$\Omega_{\pm} = \frac{\partial \mathcal{M}}{\partial J} = \frac{4\pi J}{\mathcal{M}\mathcal{A}_{\pm}}. \quad (63)$$

The Smarr-Gibbs-Duhem relation holds for Kerr BH as

$$\frac{\mathcal{M}}{2} = T_{\pm}\mathcal{S}_{\pm} + J\Omega_{\pm}. \quad (64)$$

It can be seen that both First law of BH thermodynamics and Smarr-Gibbs-Duhem relation holds for Kerr BH.

3.2 Christodoulou's Irreducible Mass for Kerr-Taub-NUT Spacetime:

Analogously, we can define the irreducible mass for Kerr-Taub-NUT black hole:

$$\mathcal{M}_{irr,\pm} = \sqrt{\frac{\mathcal{A}_{\pm}}{16\pi}} = \frac{\sqrt{r_{\pm}^2 + a^2 + n^2}}{2}. \quad (65)$$

Again, the area and angular velocity can be expressed as in terms of $\mathcal{M}_{irr\pm}$:

$$\mathcal{A}_{\pm} = 16\pi(\mathcal{M}_{irr,\pm})^2. \quad (66)$$

and

$$\Omega_{\pm} = \frac{a}{r_{\pm}^2 + a^2 + n^2} = \frac{a}{4(\mathcal{M}_{irr,\pm})^2} . \quad (67)$$

Therefore the product of the irreducible mass of \mathcal{H}^{\pm} for Kerr-Taub-NUT space-time:

$$\mathcal{M}_{irr,+}\mathcal{M}_{irr,-} = \sqrt{J^2 + n^2(\mathcal{M}^2 + n^2)} . \quad (68)$$

It is noteworthy that the product of irreducible mass of \mathcal{H}^{\pm} for Kerr-Taub-NUT black hole, depends on mass and therefore is not *universal*.

The Christodoulou-Ruffini mass formula for Kerr-Taub-NUT spacetime in term of its irreducible mass and its angular momentum J and NUT parameter n is given by

$$\mathcal{M}^2 = \left[(\mathcal{M}_{irr,\pm})^2 + \frac{J^2}{4(\mathcal{M}_{irr,\pm})^2} - n^2 \left(1 - \frac{n^2}{4(\mathcal{M}_{irr,\pm})^2} \right) \right] \times \left(1 - \frac{n^2}{4(\mathcal{M}_{irr,\pm})^2} \right)^{-1} . \quad (69)$$

When the NUT parameter goes to zero we get the mass formula for Kerr space-time:

$$\mathcal{M}^2 = (\mathcal{M}_{irr,\pm})^2 + \frac{J^2}{4(\mathcal{M}_{irr,\pm})^2} . \quad (70)$$

4 Conclusions:

In this work, we have studied both the inner horizon and outer horizon thermodynamics for Taub-NUT and Kerr-Taub-NUT BH. We also computed various thermodynamic products for these BHs. Due to the presence of the NUT parameter, we have found that the thermodynamic products have no nice quantization features. It has been also shown that the first law of BH thermodynamics does not hold for Taub-NUT and Kerr-Taub-NUT BH. We have also derived the mass formula and Christodoulou-Ruffini mass formula of the above mentioned BHs. We have also proved that Smarr-Gibbs-Duhem relation does not hold for both Taub-NUT and Kerr-Taub-NUT BH due to the presence of NUT parameter. We have compared these properties with the RN BH and Kerr BH. It is shown that such features are unlikely in the RN BH and Kerr BH. It would be interesting to investigate the various thermodynamic products due to the contribution from Dirac-Misner strings[38]. Similar studies for Kerr-Newman-Taub-NUT BH could be found in [39]

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